Performance Evaluation and Networks

Discrete time Markov Chains (MC)
Tools for analysis

How to decide irreducibility ? aperiodicity ?

Transition graph structure:

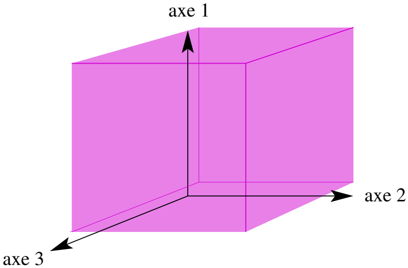
- Computing strongly connected comp and acyclic quotient graph: computable in general (depends on the chain description if nb states ∞), linear in time and space (if finite nb states) → algos based on DFS (Tarjan 1972, Kosaraju 1978)
- Computing the period: computable in general (depends on the chain description if nb states ∞), linear in time and space (if finite nb states) → algo based on graph searching (Denardo 1977)

How to decide recurrence ?

Definition (Face-homogeneous HMC over $\mathbb{N}^d$ with unit jumps)

For all $\Lambda \subseteq \{1,\ldots,d\}$, HMC (X_n) space homogeneous over the face $\mathbb{N}_\Lambda \stackrel{def}{=} \{x=(x_1,\ldots,x_d)\in\mathbb{N}^d | \forall\lambda\in\Lambda, x_\lambda=0,\ \forall\lambda\notin\Lambda, x_\lambda>0\}$ such that $\forall\Delta\in\{-1,0,+1\}^d, \forall x\in\mathbb{N}_\Lambda$, proba to jump from x to $x+\Delta$ is $p(\Lambda,\Delta)$, with $\sum_{\Delta\in\{-1,0,+1\}^d} p(\Lambda,\Delta)=1$.

face $\mathbb{N}_\Lambda$ for $\Lambda=\{1,2,3\}$



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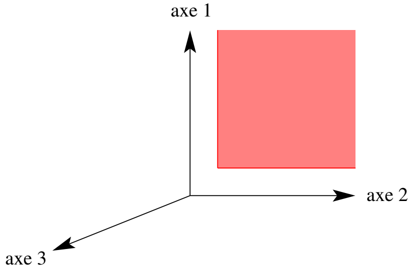
Here we choose coordinates 1 and 2, so they are stric

face $\mathbb{N}_\Lambda$ for $\Lambda = \{1,2\}$



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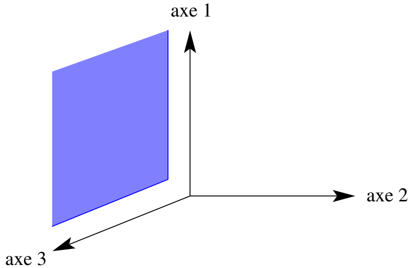
face $\mathbb{N}_\Lambda$ for $\Lambda = \{1, 3\}$



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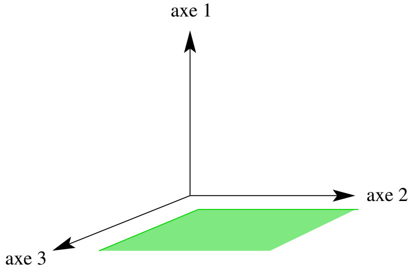

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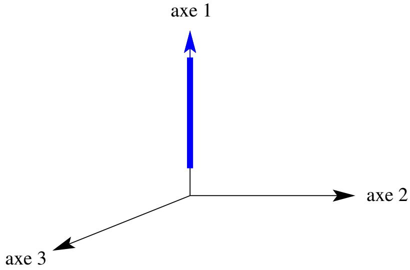
face $\mathbb{N}_\Lambda$ for $\Lambda=\{1\}$



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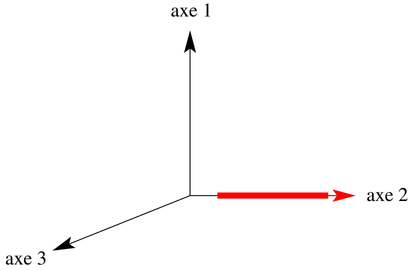
face $\mathbb{N}_\Lambda$ for $\Lambda=\{2\}$



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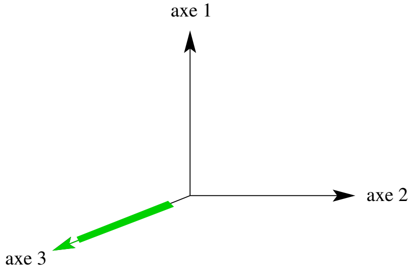
face $\mathbb{N}_\Lambda$ for $\Lambda = \{3\}$



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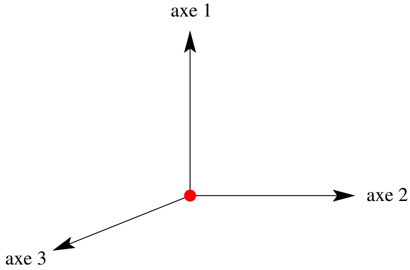
face $\mathbb{N}_\Lambda$ for $\Lambda = \emptyset$



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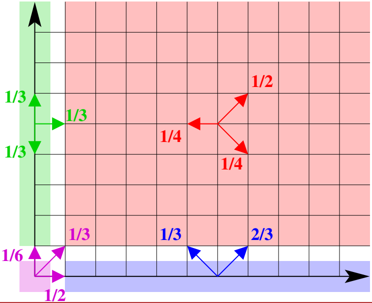
HMC
face homogeneous
with unit jumps

Ex here: if $\Lambda=\{1,2\}$ et $\Delta=(+1,+1)$
$p(\Lambda,\Delta)=1/2$



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Theorem (Gamarnik 2002)

Deciding for any d *if HMC face-homogeneous over* $\mathbb{N}^d$ *with unit jumps is positive recurrent, is undecidable.*

Theorem (Malyshev 1972, Menshikov 1974, Ignatyuk 1993)

Deciding for fixed $d \in \{1,2,3,4\}$ *if HMC face-homogeneous over* $\mathbb{N}^d$ *with unit jumps is positive recurrent, is decidable (open for fixed* $d \geq 5$*).*

How to decide recurrence ?

Useful first step: check irreducibility.

Checking recurrence:

- by returning to the definition (e.g. explicit value of $\mathbb{P}_i(T_i < \infty)$)
- by the potential matrix criterium (nature of $\sum_{n \geq 0} p_{ii}(n)$)

Checking positive recurrence:

- if finite nb states, obvious: yes iff irreducible
- by returning to the definition (e.g. explicit computation of $\mathbb{E}_i(T_i)$)
- by searching a invariant distribution (search an inv measure & check at the end that $\sum_i \pi_i < \infty$),
- by the use of super/sub-martingales.

bcs if an invariant distr exists, then we kr

Martingales: definitions

This is a very useful tools in probability, especially for Markov chain

Cond expectation of Y real r.v. with respect to r.v. $X_n, \ldots, X_0$:

$$\mathbb{E}(Y|X_n,\ldots,X_0) \overset{\text{def}}{=} \sum_{i_0,\ldots,i_n \in E} \mathbb{E}(Y|X_n = i_n,\ldots,X_0 = i_0)\mathbb{1}_{X_n=i_n,\ldots,X_0=i_0}$$ ⚠ r.v.

Definition (Martingale with respect to process $(X_n)_{n\in\mathbb{N}}$)

Process $(M_n)_{n\in\mathbb{N}}$ with real values martingale with respect to Process $(X_n)_{n\in\mathbb{N}}$ with values in E if: $\forall n \in \mathbb{N}, \mathbb{E}|M_n| < \infty$ and $\mathbb{E}(M_{n+1}|X_n,\ldots,X_0) = M_n$. In this case, $\forall n \in \mathbb{N}, \mathbb{E}(M_n) = \mathbb{E}(M_0)$.

In practice: usually $M_n \overset{\text{def}}{=} f(X_n,\ldots,X_0)$, or even $f(X_n)$, then check if $\forall i_0,\ldots,i_n \in E, \mathbb{E}(M_{n+1}|X_n = i_n,\ldots,X_0 = i_0) = f(i_n,\ldots,i_0)$.

Example: (X_n) symmetric walk over $\mathbb{Z}$, $M_n = f(X_n)$ with $f(i) = i$

Variants: sub-/super-martingale if $\forall n \in \mathbb{N}, \mathbb{E}(M_{n+1}|X_n,\ldots,X_0) \geq M_n$ (resp $\leq$) and $\mathbb{E}|M_n| < \infty$

Martingales: stopping time theorem

Theorem (Doob's stopping theorem/ optional stopping theorem)

Let (M_n) martingale (resp. sub-/super-) for (X_n) and T stopping time for (X_n). If at least one of the next conditions is true:

1. $T \leq N$ *a.s. where* $N \in \mathbb{N}$
2. $T < \infty$ *and* $\forall n \in \mathbb{N}, |M_n| \leq C$ *a.s. where* $C \in \mathbb{R}_+$
3. $\mathbb{E}(T) < \infty$ *and* $\forall n \in \mathbb{N}, |M_{n+1} - M_n| \leq C$ *a.s. where* $C \in \mathbb{R}_+$

Then $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ *(resp.* $\geq / \leq$*).*

Applications: (X_n) symmetric walk over $\mathbb{Z}$, $0 \leq i \leq N$, let $T = \tau_{\{0,N\}}$ absorption time by 0 or N

- Proba of absorption by N:
- Mean absorption time:

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- Proba of absorption by N: $M_n = X_n \Rightarrow \mathbb{P}_i(X_T = N) = i/N$
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- Proba of absorption by N: $M_n = X_n \Rightarrow \mathbb{P}_i(X_T = N) = i/N$
- Mean absorption time: $M_n = X_n^2 - 1 \Rightarrow \mathbb{E}_i(T) = i(N-i)$

Martingales: Foster's theorem (I)

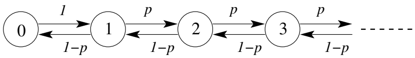
Theorem (one CS of positive recurrence - Foster 1953)

Let (X_n) HMC irred with values in E, if there exists $h : E \to \mathbb{R}_+$, F fini $\subseteq E$, $\varepsilon > 0$ such that:

- $\forall i \in F, \mathbb{E}_i(h(X_1)) = \sum_{j\in E} p_{ij} h(j) < \infty$, *and*
- $\forall i \notin F, \mathbb{E}_i(h(X_1) - h(X_0)) = \sum_{j\in E} p_{ij} h(j) - h(i) \leq -\varepsilon$

Then the chain is positive recurrent and $\forall i \in F, \mathbb{E}_i(T_F) \leq h(i)/\varepsilon$.

Example: biaised walk over $\mathbb{N}$ with $p < 1/2$



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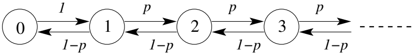

$\rightarrow$ positive recurrent: take $F = \{0\}$ and $h(i) = i$

Martingales: Foster's theorem (II)

Theorem (one CS of non positive recurrence - Tweedie 1976)

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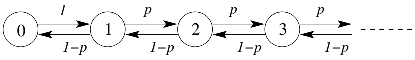
- $\forall i \notin F, \mathbb{E}_i(h(X_1) - h(X_0)) = \sum_{j\in E} p_{ij} h(j) - h(i) \geq 0$
- $\exists i_0 \notin F, h(i_0) > \max_{i\in F} h(i)$

Then the chain is not positive recurrent and $\mathbb{E}_{i_0}(T_F) = +\infty$.

Example: biaised walk over $\mathbb{N}$ with $p \geq 1/2$



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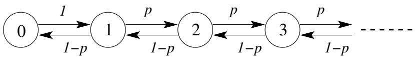


$\rightarrow$ not positive recurrent: take $F = \{0\}$ and $h(i) = i$ or $h(i) = \mathbb{1}_{\geq 1}(i)$

Invariant distribution: computation techniques

- Solve directly the linear system $\pi P = \pi$ with unknown $(\pi_i)_{i \in E}$ (combine/substitute, Gauss' pivot, Cramer's formulas ...).
- Introduce new linear equations using *flow* reasoning, to simplify the system solving.
- Pull out of the hat a good candidate, inject it in the linear system to check if it works, adjust its parameters if necessary.

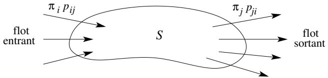
Invariant distribution: flows

Proposition ("flow" vision of invariance)

Associate with distrib $\pi = (\pi_i)_{i\in E}$ *the flow* $f_{ij} \overset{def}{=} \pi_i p_{ij}$ *from* i *to* j *for each edge* ij *in the transition graph. Then* π *inv distrib iff* f *satisfies Kirchoff's 1st law (preservation of the total flow at each state).*

Proposition (Flow relations in the stationary regime)

Let π invariant *distrib and* $S \subseteq E$*, then:* $\sum_{\substack{i\notin S\\ j\in S}} \pi_i p_{ij} = \sum_{\substack{j\in S\\ i\notin S}} \pi_j p_{ji}$



Example: reversible Markov chains

Modeling steps with discrete time HMC

1. Define the space of states, list the states if possible
2. For each state, list events that may occur
3. Check whether the dynamics is Markovian, homegeneous for time and/or space

Examples: some models based on discrete time M/M/1 queues